Coasting Arcs in Optimal Power-Limited Rocket Flight

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One of the problems treated in Ref. 1 is that of optimal thrust control for power-limited flight. It is shown there that operation at maximum propulsive power is optimal, provided limitation is on power alone. In effect, it can be shown that operation at maximum power is optimal, provided the thrust is nonzero, but that coasting flight may be optimal under special, highly restricted, circumstances. In the ensuing analysis, the nomenclature of Ref. 1 is used.

The optimal thrust-magnitude program may be found from the maximum principle—namely, power α and mass-flow rate β must be such as to maximize

$$L = (\sqrt{\alpha\beta/m})\sqrt{\lambda_u^2 + \lambda_v^2} - \lambda_m\beta \quad (1)$$

where

$$0 \leq \alpha \leq \alpha_{max} \quad (2)$$

and β is not assumed bounded. With respect to α, provided $\beta \neq 0$, it is required that

$$\alpha = \alpha_{max} \quad (3)$$

With respect to β, two cases may arise, depending on the adjoint variable λ_m, associated with mass variation. These are illustrated in Fig. 1.

Thus, if

$$\lambda_m \leq 0 \quad (4)$$

L does not possess a stationary maximum and the optimal choice of β is the largest possible value. If

$$\lambda_m > 0 \quad (5)$$

there always exists a stationary maximum of L, and the optimal choice of β is given by

$$\partial L/\partial\beta = 0 \quad (6)$$

A solution of Eq. (6) corresponding to coasting flight—i.e.,

$$\beta \equiv 0 \quad (7)$$

can occur only if

$$\lambda_u = \lambda_v \equiv 0 \quad (8)$$

and the latter condition implies

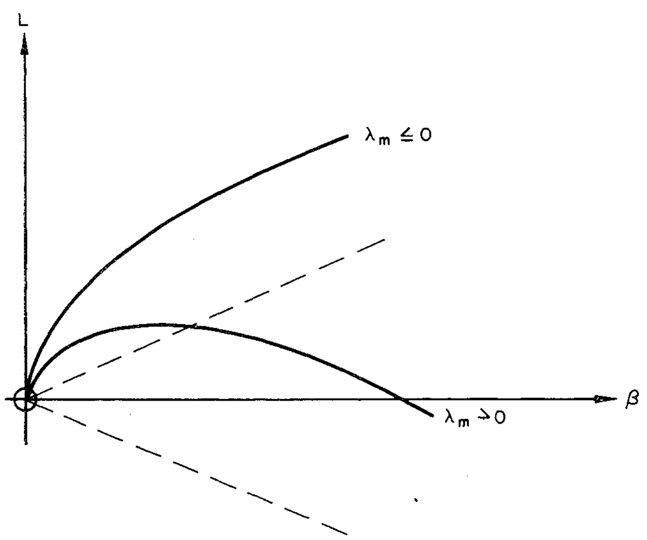

Fig. 1.

$$\lambda_x = \lambda_y \equiv 0 \quad (9)$$

$$\lambda_m = \text{constant} \quad (10)$$

as can be seen by inspection of the adjoint equations, Eq. (5.45) of Ref. 1.

Since the adjoint variables are the partial derivatives of the payoff with respect to the state variables—i.e.,

$$\lambda_q = -\partial J/\partial q \quad (11)$$

where λ_q is the adjoint variable associated with state variable q, and J is the value of the quantity to be minimized (from state q to the end state)—it can be seen that coasting flight is optimal only if the payoff is insensitive to first-order changes in position and velocity. It is doubtful that many meaningful problems fall into this category. However, to illustrate the possibility, consider the following trivial problem.

It is desired to transfer maximum mass in a field-free environment between equal velocities without restriction on position—i.e.,

$$\min(-m_f) \quad (12)$$

with end conditions

$$\left.\begin{array}{l} t = 0:\ m = m_0,\quad x = y = 0,\quad u = u_0,\quad v = 0 \\ t = t_f:\ u = u_0,\quad v = 0 \end{array}\right\} \quad (13)$$

Of course, the answer is $m_f = m_0$ and the trajectory is solely coasting. Conditions (8) to (10) are clearly met. Since for field-free flight

$$\lambda_x = \text{const.},\quad \lambda_y = \text{const.} \quad (14)$$

and by Eq. (11)

$$\lambda_x = \lambda_y = 0 \quad (15)$$

and also

$$\lambda_u = \lambda_v = 0 \quad (16)$$

Furthermore, by Eq. (11)

$$\lambda_m = -\partial(-m_f)/\partial m_0 = 1 > 0 \quad (17)$$

so that all conditions are met.

Reference

[1] Leitmann, G., *Optimization Techniques*, Chap. 5, pp. 182f; Academic Press, New York, 1962.

A Matrix Formulation of the Transverse Structural Influence Coefficients of an Axially Loaded Timoshenko Beam

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A set of structural influence coefficients (SICs) is required to carry out a vibration analysis by collocation methods. For applications to many missile configurations, the structure often may be assumed to be statically determinate and idealized as a Timoshenko beam—i.e., both bending and shearing deformations are considered. In addition, the effect of axial thrust must be considered, for, although the thrust is usually much less than the buckling load, the small changes in the transverse-vibration frequencies due to thrust are important in the design of filters in the guidance-and-control system. A routine formulation of the SICs may be obtained from the methods of elastic energy. We follow the method of Ogness.[1]

Consider a statically determinate system composed of N structural elements. The total strain energy in bending and shearing deformations is given by

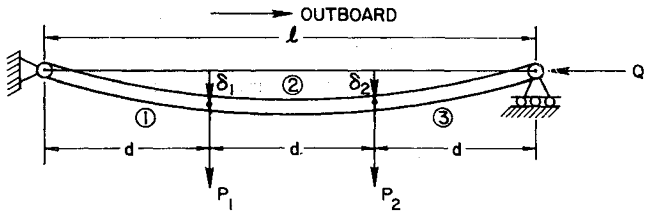

FIG. 1. Simply supported Timoshenko beam.

$$2U = \sum_{n=1}^{N} \int_0^{l_n} (M^2/EI + V^2/kAG)\, dx \tag{1}$$

where x is measured along the length of each element, M is the moment, V is the shear, and EI and kAG are the bending and shear stiffnesses, respectively (k is the cross-sectional area shape factor). With certain reasonably accurate approximations, the integral can be evaluated, and Eq. (1) replaced by a matrix expression, as will be shown.

We assume a linear variation in moment along the length of each element

$$M = M_i(1 - x/l) + M_0(x/l) \tag{2}$$

where subscripts i and 0 refer, respectively, to the inboard and outboard ends of the structural element and the origin of the x coordinate is taken at the inboard end. We next assume a constant shear between panel points:

$$V = \text{constant} \tag{3}$$

that is, we neglect the effect of taper in carrying shear as a transverse component of the bending stresses. Finally, we assume linear relationships in the reciprocals of the stiffnesses:

$$1/EI = (1/E_iI_i)(1 - x/l) + (1/E_0I_0)(x/l) \tag{4}$$

$$1/kAG = (1/k_iA_iG_i)(1 - x/l) + (1/k_0A_0G_0)(x/l) \tag{5}$$

This form of approximation to the stiffnesses also allows inclusion of variations in material properties—e.g., with temperature. Substituting Eqs. (2) through (5) into Eq. (1) permits evaluation of the integral and leads to the elastic energy:

$$2U = \sum_{n=1}^{N} [(K_{Bi}M_i + K_{Bi0}M_0)M_i + (K_{B0}M_0 + K_{Bi0}M_i)M_0 + K_SV^2] \tag{6}$$

where the flexibility constants are

$$K_{Bi} = (l/12)(3/E_iI_i + 1/E_0I_0) \tag{7}$$
$$K_{B0} = (l/12)(3/E_0I_0 + 1/E_iI_i) \tag{8}$$
$$K_{Bi0} = (1/4)(K_{Bi} + K_{B0}) \tag{9}$$
$$K_S = (l/2)(1/k_iA_iG_i + 1/k_0A_0G_0) \tag{10}$$

Eq. (6) may now be cast into matrix form. (In the following, column, diagonal, and rectangular matrices are denoted by { }, ⌈ ⌋, and [ ], respectively.)

$$2U = (\{M_i\}^T\lceil K_{Bi}\rfloor + \{M_0\}^T\lceil K_{Bi0}\rfloor)\{M_i\} + (\{M_0\}^T\lceil K_{B0}\rfloor + \{M_i\}^T\lceil K_{Bi0}\rfloor)\{M_0\} + \{V\}^T\lceil K_S\rfloor\{V\} \tag{11}$$

The moments and shear in terms of the external loading are expressed as

$$\{M_i\} = [M_i/P]\{P\} + [M_i/\delta]\{\delta\} \tag{12}$$

$$\{M_0\} = [M_0/P]\{P\} + [M_0/\delta]\{\delta\} \tag{13}$$

$$\{V\} = [V/P]\{P\} + [V/\delta]\{\delta\} \tag{14}$$

where $\{P\}$ is the set of transverse loads and $\{\delta\}$ is the set of deflections under the transverse loads. The matrices of coefficients of $\{P\}$ in Eqs. (12) through (14), therefore, are the quantities necessary to specify the contribution of each transverse load to each internal load; the matrices of coefficients of $\{\delta\}$ are the quantities necessary to specify the contribution of the axial loads to each internal load. The following example illustrates the generation of all of the necessary elements of the coefficient matrices.

We are now in a position to calculate the transverse deflections by Castigliano's theorem, since the elastic energy is related to the transverse loads once Eqs. (11) through (14) are combined.

$$\{\delta\} = \{\partial U/\partial P\} \tag{15a}$$

$$= [A]\{M_i\} + [B]\{M_0\} + [C]\{V\} \tag{15b}$$

where

$$[A] = [M_i/P]^T\lceil K_{Bi}\rfloor + [M_0/P]^T\lceil K_{Bi0}\rfloor \tag{16}$$

$$[B] = [M_0/P]^T\lceil K_{B0}\rfloor + [M_i/P]^T\lceil K_{Bi0}\rfloor \tag{17}$$

$$[C] = [V/P]^T\lceil K_S\rfloor \tag{18}$$

Substituting Eqs. (12) through (14) into Eq. (15b) permits solution for $\{\delta\}$ in terms of $\{P\}$, which is the defining relationship for the SICs

$$\{\delta\} = [a]\{P\} \tag{19}$$

where the SICs are

$$[a] = (\lceil I\rfloor - [b])^{-1}[a_0] \tag{20}$$

$\lceil I\rfloor$ is the unit matrix, $[a_0]$ is the set of SICs without axial load,

$$[a_0] = [A][M_i/P] + [B][M_0/P] + [C][V/P] \tag{21}$$

and $(\lceil I\rfloor - [b])^{-1}$ gives the amplification due to axial load, where

$$[b] = [A][M_i/\delta] + [B][M_0/\delta] + [C][V/\delta] \tag{22}$$

Buckling occurs when the matrix $(\lceil I\rfloor - [b])$ becomes singular.

The calculation of the various coefficient matrices with the uniform simply supported beam shown in Fig. 1 will now be illustrated. The uniform beam has the following third-order flexibility-constant matrices

$$\left.\begin{aligned} \lceil K_{Bi}\rfloor &= \lceil K_{B0}\rfloor = 2\lceil K_{Bi0}\rfloor = (d/3EI)\lceil I\rfloor \\ \lceil K_S\rfloor &= (d/kAG)\lceil I\rfloor \end{aligned}\right\} \tag{23}$$

To form the moment matrices due to transverse loading, we note the moments are $M_{i1} = 0$, $M_{i2} = M_{01} = (2/3)P_1d + (1/3)P_2d$, $M_{i3} = M_{02} = (1/3)P_1d + (2/3)P_2d$, and $M_{03} = 0$. Hence,

$$\left.\begin{aligned} [M_i/P] &= (d/3)\begin{bmatrix} 0 & 0 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \\ [M_0/P] &= (d/3)\begin{bmatrix} 2 & 1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{24}$$

The shears due to transverse loading are $V_1 = (2/3)P_1 + (1/3)P_2$, $V_2 = -(1/3)P_1 + (1/3)P_2$, $V_3 = -(1/3)P_1 - (2/3)P_2$, so that

$$[V/P] = (1/3)\begin{bmatrix} 2 & 1 \\ -1 & 1 \\ -1 & -2 \end{bmatrix} \tag{25}$$

To form the moment matrices due to axial load, we note $M_{i1} = 0$, $M_{i2} = M_{01} = Q\delta_1$, $M_{i3} = M_{02} = Q\delta_2$, and $M_{03} = 0$, so that

$$\left.\begin{aligned} [M_i/\delta] &= Q\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \\ [M_0/\delta] &= Q\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}\right\} \tag{26}$$

We note that the original assumption of a linear variation in moment along the length of each element implies a linear deflection curve between control points when calculating the effect of axial load. This approximation is considered adequate in dealing with axial loads much less than the buckling load, and is undoubtedly adequate for calculating the buckling load if the structure is divided into a reasonably large number of elements (say, five or six). The shear due to the axial load is found from the transverse component of the axial load along the deflected beam.[2] Hence, the matrix $[V/\delta]$ is a differentiation matrix that leads to the product of the *average* slope of each element (corresponding to the linear deflection curve assumption) and the axial load. The shears due to axial load are, therefore, $V_1 = Q\delta_1/d$, $V_2 = Q(\delta_2 - \delta_1)/d$, $V_3 = -Q\delta_2/d$, so that

$$[V/\delta] = (Q/d)\begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \quad (27)$$

Noting that $d = l/3$ and carrying out the calculations of Eqs. (21) and (22) leads to

$$[a_0] = (l^3/486EI)\begin{bmatrix} 8 & 7 \\ 7 & 8 \end{bmatrix} + (l/9kAG)\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad (28)$$

$$[b] = (Ql^2/54EI)\begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix} + (Q/kAG)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (29)$$

The lower load at which $([I] - [b])$ becomes singular is

$$Q_{cr} = (10.8EI/l^2)/(1 + 10.8EI/l^2kAG) \quad (30)$$

and agrees within 10 percent with the exact result given by Timoshenko[2] (the numerical coefficient should be π^2). The approximations with regard to the interaction of the axial load and the shape of the deflection curve are surprisingly accurate considering that the example was divided into only three structural elements.

The preceding derivation leads to an extremely useful expression for the SICs of arbitrary structural configurations, provided they can be approximated as statically determinate systems of simple beams. The value of the method lies both in its ability to account for nonuniformity in the stiffness and in the routine manner in which the elements of the moment- and shear-coefficient matrices can be assembled. The extension of the foregoing method to obtain SICs for torsional and axial deformations is straightforward and leads to terms similar to those shown for the shear deformation.

References

[1] Ogness, A. M., *A Specific Technique for Applying Castigliano's Theorem in the Solution of Redundant Structures*, presented at IAS Specialist Meeting, Los Angeles, Calif., 1951.

[2] Timoshenko, S., *Theory of Elastic Stability*, p. 139; McGraw-Hill Book Co., Inc., New York, 1936.

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